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Characterizing the sustainability problem in an exhaustible resource model

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Abstract

We provide a general condition under which consumption can be sustained indefinitely bounded away from zero in the continuous time Dasgupta–Heal–Solow–Stiglitz model, by letting augmentable capital substitute for a non-renewable resource. The assumptions made on the production function are mild, thus generalizing previous work. By showing that Hartwick's rule minimizes the required resource input per unit of capital accumulation, and integrating the required resource input with respect to capital, we obtain a complete technological characterization without reference to the time path. We also use the characterization result to establish general existence of a maximin path.

JEL classification: O10; Q32

Keywords: Sustainability; Maximin

1. Introduction

This paper provides a general condition under which consumption can be sustained indefinitely bounded away from zero in the continuous time Dasgupta–Heal–Solow–Stiglitz (DHSS) [8,19,20] model, by letting accumulated augmentable capital substitute for depleted exhaustible

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resource. The assumptions made on the production function are mild: both the stock of capital and the flow of resource input are essential, and the production function is twice continuously differentiable, monotonically increasing in both inputs and concave, and exhibits strictly diminishing marginal returns with respect to resource input.

We show that sustainability is equivalent to the initial resource stock being larger than the cumulative resource input in a *Hartwick path* where, following Hartwick's investment rule, capital accumulation (evaluated at competitive prices) exactly compensates for resource use, for an arbitrarily small constant consumption level. The key observation is that obeying Hartwick's rule minimizes the required resource input per unit of capital accumulation if consumption is to be sustained at a positive and constant level.[1] By integrating the required resource input with respect to capital we obtain a complete technological characterization of sustainability without reference to the time path.

Furthermore, we establish general existence of a maximin path by showing that the set of constant consumption levels, for which the cumulative resource input in the Hartwick path does not exceed the initial resource stock, is bounded above and contains its least upper bound. Finally, we show that the maximin Hartwick path exhausts the resource and thus is efficient if and only if the following scenario does not arise: there is a maximal finite cumulative resource input that can be attained by a Hartwick path, and this maximum falls short of the initial resource stock.[2]

This paper thus completes a research agenda initiated by Solow [19]. He let output be a Cobb–Douglas function of capital and resource input, and showed that an efficient and egalitarian maximin path with positive consumption exists if and only if the elasticity of output with respect to capital, a, exceeds that with respect to resource input, b. Moreover, if this condition is *not* satisfied, the greatest lower bound for consumption is zero, so that no positive level of consumption can be sustained indefinitely and any path solves the maximin problem. Together, these observations show that in the Cobb–Douglas case, (i) the solution of the sustainability problem depends on whether $a > b$ and (ii) a maximin path always exists.

Our sustainability characterization result (Theorem 1) and our maximin existence and efficiency result (Theorem 2) extend the two results of Solow to a very general class of production functions. The generalization is useful as it allows one to interpret the two inputs in a more general way. For instance, one could consider the augmentable capital to also encompass human capital and technology, which may not be faced with strictly diminishing returns. Also, the resource might not only be interpreted as fossil fuels since, in the very long run, the atmosphere's cumulative capacity for absorbing CO_2 (without causing serious climate change) is a non-renewable and exhaustible resource.

We follow Solow by taking an indirect route to establishing existence of a maximin path by positing a class of candidate paths, since one of the standard assumptions of existence theory—pointwise boundedness of the relevant variables—is not satisfied: the flow of resource input has no a priori upper bound, only its integral is bounded. Hence, maximin existence is a non-trivial result, and the analysis cannot be based on necessary conditions from a problem of minimizing the integral of resource input.

The sustainability problem has relevance even if one does not ascribe to maximin as an extreme egalitarian criterion. In particular, a non-trivial sustainable path in the DHSS model matters also for the criteria of undiscounted utilitarianism [9, Sect. 10.3], sustainable discounted

[1] This was suggested already by Buchholz [4, pp. 69–70] (see also [14, p. 25]).

[2] To show that, for any initial resource stock, there is a Hartwick path with a sufficiently high constant consumption level that exhausts the resource appears to require further assumptions.

utilitarianism [2, Sect. 5], and (extended) rank-discounted utilitarianism [21, Sect. 6.2]: in the former case no optimal path exists while in the latter cases all paths are equally bad if positive consumption cannot be sustained.

The paper is organized as follows. In Section 2 we introduce the DHSS model with its assumptions. In Section 3 we present our main results, which are based on propositions that are proven in Section 4. The proofs of the propositions in turn use several lemmas, the proofs of which are included in online appendix A.

2. Preliminaries

Denote by k the stock of an augmentable capital good (which is assumed to be non-depreciating) and by r the flow of an exhaustible resource input. Denote by $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ the production function for the capital/consumption good, employing k and r as inputs. The output $F(k,r)$ is used to provide a flow of consumption, c, or to augment the capital stock through a flow of net investment, $\dot{k}$. Throughout we impose three assumptions on F (where subscripts denote partial derivatives):

Assumption 1 *(**A1**).* $F(0,r) = F(k,0) = 0$ for $k \in \mathbb{R}_+$ and $r \in \mathbb{R}_+$.

Assumption 2 *(**A2**).* F is continuous, concave and nondecreasing on $\mathbb{R}^2_+$.

Assumption 3 *(**A3**).* F is twice continuously differentiable on $\mathbb{R}^2_{++}$, with $F_1(k,r) > 0$, $F_2(k,r) > 0$, and $F_{22}(k,r) < 0$ for $(k,r) \in \mathbb{R}^2_{++}$.

Let $(k_0, m_0) \in \mathbb{R}^2_{++}$ be a vector of initial stocks of capital and resource. A *path* from k_0 is a triplet of functions $(c(t), k(t), r(t))$, with $c(\cdot) : [0,\infty) \to \mathbb{R}_+$, $k(\cdot) : [0,\infty) \to \mathbb{R}_+$ and $r(\cdot) : [0,\infty) \to \mathbb{R}_+$, where $k(t)$ is differentiable and $(c(t), r(t))$ are continuous, and where

$$\dot{k}(t) = F\big(k(t), r(t)\big) - c(t); \qquad k(0) = k_0. \tag{1}$$

Write $m(\cdot) : [0,\infty) \to \mathbb{R}$ for the associated function of remaining resource stock:

$$m(t) = m_0 - \int_0^t r(\tau)\,d\tau \quad \text{for } t \geqslant 0.$$

A *feasible path* $(c(t), k(t), r(t))$ from (k_0, m_0) is a path from k_0 satisfying $m(t) \geqslant 0$ for all $t \geqslant 0$. Note that along any feasible path $(c(t), k(t), r(t))$ from (k_0, m_0), both $k(t)$ and $m(t)$ are continuously differentiable functions of t. A feasible path $(c(t), k(t), r(t))$ from (k_0, m_0) is *resource exhausting* if $\int_0^\infty r(t)\,dt = m_0$, and *efficient* if there is no feasible path $(c'(t), k'(t), r'(t))$ from (k_0, m_0) with $c'(t) \geqslant c(t)$ for all $t \geqslant 0$ and $c'(\tau) > c(\tau)$ for some $\tau \geqslant 0$.[3] A triplet $(c,k,r) \in \mathbb{R}^3_{++}$ satisfies *Hartwick's* reinvestment *rule* [12,11] if

$$F(k,r) - c = F_2(k,r)r. \tag{HaR}$$

[3] Usually we employ the strict inequality for an interval of time, because in continuous time spikes do not matter. Here, however, $c(t)$ is continuous, so that we can use this equivalent definition.

A path $(c(t), k(t), r(t))$ from k_0 is *egalitarian* if there is $c \geqslant 0$ such that $c(t) = c$ for all $t \geqslant 0$. A feasible path $(c(t), k(t), r(t))$ from (k_0, m_0) is a *maximin path* if

$$\inf_{t \geqslant 0} c(t) \geqslant \inf_{t \geqslant 0} c'(t) \tag{2}$$

for every feasible path $(c'(t), k'(t), r'(t))$ from (k_0, m_0). Refer to $\inf_{t \geqslant 0} c(t)$ as the *maximin value* if $(c(t), k(t), r(t))$ from (k_0, m_0) is a maximin path, where it follows from the definition of a maximin path that all maximin paths have the same maximin value. A maximin path is *non-trivial* if the maximin value is positive.

Define the set of positive sustainable consumption levels as:

$$C(k_0, m_0) = \big\{c \in \mathbb{R}_{++}\colon \text{there is a feasible path } \big(c(t), k(t), r(t)\big) \text{ from } (k_0, m_0) \text{ with } c(t) \geqslant c \text{ for } t \geqslant 0\big\}.$$

Assumptions **A1**–**A3** do *not* imply that this set is non-empty. In particular, if

$$F(k, r) = k^a r^b \quad \text{for } (k, r) \in \mathbb{R}^2_+, \text{ with } a > 0, b > 0 \text{ and } a + b \leqslant 1, \tag{3}$$

then assumptions **A1**–**A3** are clearly satisfied. However, as shown by Solow [19, Sect. 8 & App. B], the set $C(k_0, m_0)$ is non-empty if and only if $a > b$.

3. Main results

In order to relate Hartwick's rule to the minimization of resource input and thus provide technological conditions for sustainability we will make use of the *resource requirement functions* established in the following lemma by considering the problem

$$\min_{\{r\colon F(k,r) > c\}} \frac{r}{F(k, r) - c}. \tag{4}$$

The domain of these resource requirement functions is the set of consumption–capital pairs which allow for positive capital accumulation:

$$D = \big\{(c, k) \in \mathbb{R}^2_{++}\colon \text{there is } r > 0 \text{ such that } F(k, r) > c\big\}.$$

Note that D is non-empty since $(c, 1) \in D$ whenever $0 < c < F(1, 1)$. For every $k \in \mathbb{R}_{++}$, $D(k) \equiv \{c \in \mathbb{R}_{++}\colon (c, k) \in D\}$ is the non-empty interval $(0, \lim_{r \to \infty} F(k, r))$.

Lemma 1. *Assume* **A1**–**A3**. *Then* $\mathbf{p} : D \to \mathbb{R}_{++}$ *and* $\mathbf{r} : D \to \mathbb{R}_{++}$ *defined by*

$$\mathbf{p}(c, k) \equiv \min_{\{r\colon F(k,r) > c\}} \frac{r}{F(k, r) - c} \quad \textit{and}$$

$$\mathbf{r}(c, k) \equiv \operatorname*{arg\,min}_{\{r\colon F(k,r) > c\}} \frac{r}{F(k, r) - c} \quad \textit{for all } (c, k) \in D$$

are continuously differentiable single-valued functions with

$$\mathbf{p}_1(c, k) > 0 \quad \textit{and} \quad \mathbf{r}_1(c, k) > 0 \quad \textit{for all } (c, k) \in D. \tag{5}$$

Furthermore, (c, k, r) *satisfies* (HaR) *if and only if* $r = \mathbf{r}(c, k)$.

For $(c, k) \in D$, problem (4) has the following first-order condition:

$$F\big(k, \mathbf{r}(c, k)\big) - c = F_2\big(k, \mathbf{r}(c, k)\big)\mathbf{r}(c, k),$$

verifying that Hartwick's rule (cf. (HaR)) is satisfied at its minimum. Hence,

$$\frac{1}{F_2\big(k, \mathbf{r}(c,k)\big)} = \mathbf{p}(c,k) = \frac{\mathbf{r}(c,k)}{F(k, \mathbf{r}(c,k)) - c},$$

implying that $\mathbf{p}(c,k)$ can be interpreted in two ways: The left-hand side is the marginal cost of $\dot{k}$ in terms of r, while by (1) the right-hand side is the average cost of $\dot{k}$ in terms of r, keeping consumption fixed at c. Since the marginal cost of $\dot{k}$ in terms of r is increasing in r (cf. **A3**), the average cost of $\dot{k}$ in terms of r is minimized by obeying Hartwick's reinvestment rule, and $\mathbf{p}(c,k)$ is the *required resource input per unit of capital accumulation* if consumption is to be sustained at c.

If $(c,k) \in D$, then by Lemma 1 there are $k' < k$ and $c' > c$ such that $(0,c') \times (k', \infty) \in D$. Since $\mathbf{p}(c,\cdot)$ is continuous on (k', ∞), the Riemann integral $\int_k^{k''} \mathbf{p}(c,x)\,dx$ is well-defined for every $k'' > k$. Hence, we may define $\mathbf{m} : D \to \mathbb{R}_{++} \cup \{\infty\}$ by

$$\mathbf{m}(c,k) = \int_k^\infty \mathbf{p}(c,x)\,dx.$$

Since $\mathbf{p}(c,x)$ is the required resource input per unit of capital accumulation if the capital stock equals x, the function $\mathbf{m}$ determines the required cumulative resource input needed to sustain the consumption level c from the initial capital stock k. It is found by integrating $\mathbf{p}(c,\cdot)$ from k to ∞, and enables one to obtain *a technological characterization of the sustainability problem without reference to a time path*. That the upper bound of this integral is ∞ follows from the result that, for a technology satisfying assumptions **A1**–**A3**, capital inevitably has to grow to infinity if consumption is to be bounded away from zero forever (cf. Lemma 5 of Section 4).

To establish the relationship to the corresponding time path, we must show that there exists a unique solution on $[0, \infty)$ to the initial value problem

$$\dot{x}(t) = F\big(x(t), \mathbf{r}\big(c, x(t)\big)\big) - c; \qquad x(0) = k_0. \tag{6}$$

If $(c, k_0) \in D$ and problem (6) has a unique solution $k^c(t)$ for $t \in [0, \infty)$, then

$$\begin{aligned}\int_0^\infty \mathbf{r}\big(c, k^c(t)\big)\,dt &= \int_0^\infty \frac{\mathbf{r}(c, k^c(t))}{F(k^c(t), \mathbf{r}(c, k^c(t))) - c}\dot{k}^c(t)\,dt \\ &= \int_0^\infty \mathbf{p}\big(c, k^c(t)\big)\dot{k}^c(t)\,dt = \int_{k_0}^\infty \mathbf{p}(c,x)\,dx = \mathbf{m}(c, k_0)\end{aligned} \tag{7}$$

by (6) and the change of variable formula. In this case, refer to the egalitarian path $(c^c(t), k^c(t), r^c(t))$ with $c^c(t) = c > 0$ and $r^c(t) = \mathbf{r}(c, k^c(t))$ for $t \in [0, \infty)$ as a *Hartwick path* from k_0. Define the Hartwick path from k_0 with $c = 0$ as the trivial egalitarian path $(c^0(t), k^0(t), r^0(t))$ where $c^0(t) = 0$, $k^0(t) = k_0$ and $r^0(t) = 0$ for all $t \in [0, \infty)$; this path exists and is always feasible.

The basic intuition for our results is illustrated in Fig. 1: If there exists a feasible path from (k_0, m_0) that sustains consumption at or above $c > 0$ — illustrated by the dashed path in the figure — then, by (7), the Hartwick path from k_0 with $c(t) = c$ for $t \in [0, \infty)$ satisfies $\int_0^\infty r^c(t)\,dt = \int_0^\infty \mathbf{r}(c, k^c(t))\,dt = \mathbf{m}(c, k_0) \leqslant m_0$ and is thus feasible, since it minimizes resource input per unit of capital accumulation.

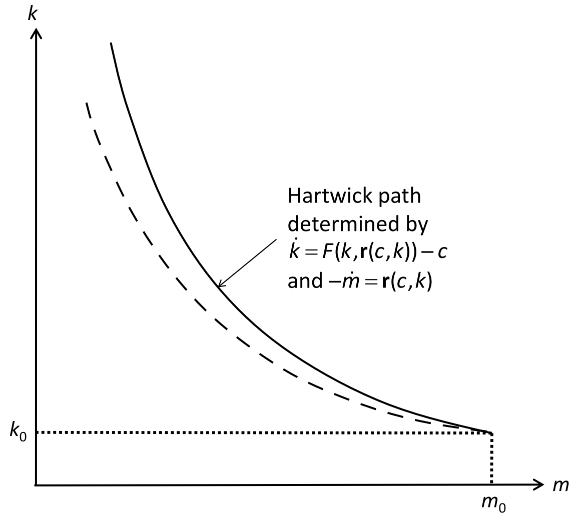

Fig. 1. Minimizing the resource depletion/capital accumulation ratio.

We can now state our *sustainability characterization result* which shows how the cumulative resource requirement function **m** can be used to formulate a necessary and sufficient condition for sustainability without making reference to a time path.

Theorem 1. *Assume* **A1**–**A3***, and let* $(k_0, m_0) \in \mathbb{R}^2_{++}$ *be given. Then* $C(k_0, m_0)$ *is non-empty if and only if* $\inf_{c \in D(k)} \mathbf{m}(c, k) < m_0$.

Remark 1. The Cass–Mitra [7] integral criterion characterization of sustainability also translates information about time paths to information about the technology. However, while the present characterization focuses directly on maintaining constant consumption (by following Hartwick's rule), the Cass–Mitra characterization focuses on behavior associated with maintaining constant output as a means to providing a consumption stream that is bounded away from zero.[4]

Our *maximin existence result* first shows that a maximin path exists for any $(k_0, m_0) \in \mathbb{R}^2_{++}$, and that the maximin value can be attained by following a Hartwick path from k_0. To describe the *maximin efficiency result* we introduce two definitions: Let $C^*(k) \equiv \{c \in D(k)\colon\ \mathbf{m}(c, k) < \infty\}$, and define $m^* : \mathbb{R}_{++} \to \mathbb{R}_+ \cup \{\infty\}$ by

$$m^*(k) = \begin{cases} \sup_{c \in C^*(k)} \mathbf{m}(c, k) & \text{if } C^*(k) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Given $(k_0, m_0) \in \mathbb{R}^2_{++}$, efficiency of the maximin Hartwick path is ensured if m_0 is smaller than or equal to $m^*(k_0)$ which may be finite or infinite. With a strict inequality the maximin Hartwick path is *regular*, which essentially means that if, along such a path, consumption is decreased on

[4] The Cass–Mitra characterization is in a discrete-time model, where (as noted by Dasgupta and Mitra [10]) Hartwick's rule does not hold for efficient and egalitarian paths. So, obeying Hartwick's rule is not a natural benchmark in that setting. Discussion of the relationship to this and other relevant literature (in particular [3,9,15,18]) can be found in [17, Sect. 2].

a small initial interval $(0, t)$, then consumption can be increased beyond t by a constant positive amount.[5]

Theorem 2. *Assume* **A1–A3***, and let* $(k_0, m_0) \in \mathbb{R}^2_{++}$ *be given. Then there exists a maximin path from* (k_0, m_0)*. Furthermore, the Hartwick path from* k_0 *that keeps consumption equal to the maximin value exists and is*

(i) *feasible and thus a maximin path,*
(ii) *efficient if and only if* $m_0 \leqslant m^*(k_0)$*,*
(iii) *a regular maximin path if* $m_0 < m^*(k_0)$*.*

The proofs of the two theorems are based on the following five propositions, which in turn are proven in the subsequent Section 4. The first proposition shows that if the value of the cumulative resource input function $\mathbf{m}$ is finite for a given consumption–capital pair (c, k_0), then the Hartwick path from k_0 with $c(t) = c$ for $t \in [0, \infty)$ exists and has a cumulative resource input that is given by $\mathbf{m}(c, k_0)$.

Proposition 1. *Assume* **A1–A3***, and let* $k_0 \in \mathbb{R}_{++}$ *be given. Assume that* $C^*(k_0) \neq \emptyset$ *and let* $c \in C^*(k_0)$*. Then there exists a Hartwick path* $(c^c(t), k^c(t), r^c(t))$ *from* k_0 *with* $c^c(t) = c$ *for all* $t \geqslant 0$*. Furthermore,* $\int_0^\infty r^c(t)\,dt = \mathbf{m}(c, k_0)$*.*

Proposition 1 implies $C(k_0, m_0) \supseteq \{c \in D(k_0){:}\ \mathbf{m}(c, k_0) \leqslant m_0\}$.

The next proposition establishes the monotonicity and continuity properties of cumulative resource input as the function $\mathbf{m}(\cdot, k_0)$ of consumption c and states that if the domain of this function is bounded, then it contains its upper bound.

Proposition 2. *Assume* **A1–A3***, and let* $k_0 \in \mathbb{R}_{++}$ *be given. Assume that* $C^*(k_0) \neq \emptyset$*. If* $c' \in C^*(k_0)$*, then* $c \in C^*(k_0)$ *for all* $c \in (0, c')$*. Furthermore,* $\mathbf{m}(\cdot, k_0)$ *is strictly increasing and continuous on* $C^*(k_0)$*. Finally, if* $c' > 0$ *and there is* $m < \infty$ *such that* $\mathbf{m}(c, k_0) \leqslant m$ *for all* $c \in (0, c')$*, then* $\mathbf{m}(c', k_0) \leqslant m$ *and thus* $c' \in C^*(k_0)$*.*

The next proposition states that cumulative resource input along any feasible path from (k_0, m_0) which sustains some strictly positive consumption level c' forever is not smaller than the cumulative resource input given by $\mathbf{m}(c', k_0)$.

Proposition 3. *Assume* **A1–A3***, and let* $(k_0, m_0) \in \mathbb{R}^2_{++}$ *be given. If there exists a feasible path* $(c(t), k(t), r(t))$ *from* (k_0, m_0) *with* $c(t) \geqslant c' > 0$ *for all* $t \geqslant 0$*, then* $\mathbf{m}(c', k_0) \leqslant m_0$*.*

Proposition 3 implies $C(k_0, m_0) \subseteq \{c \in D(k_0){:}\ \mathbf{m}(c, k_0) \leqslant m_0\}$, meaning that $\mathbf{m}$ determines *required* cumulative resource input, so that jointly with Proposition 1:

$$C(k_0, m_0) = \bigl\{c \in D(k_0){:}\ \mathbf{m}(c, k_0) \leqslant m_0\bigr\}. \tag{8}$$

[5] The formal definition of a *regular maximin path* due to Burmeister and Hammond [6] and Dixit et al. [11] will be provided in Section 4 prior to proving Proposition 5.

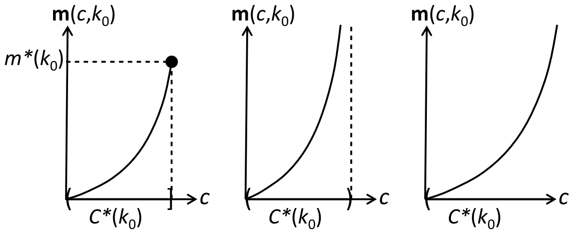

Fig. 2. The cumulative resource requirement function.

Proposition 4. *Assume* **A1–A3***, and let* $(k_0, m_0) \in \mathbb{R}^2_{++}$ *be given. Assume that* $C^*(k_0) \neq \emptyset$*. Then* $\mathbf{m}(\cdot, k_0)$ *is convex on* $C^*(k_0)$ *with range that is either equal to* $(0, \infty)$ *or equal to* $(0, m^*(k_0)]$ *where* $m^*(k_0) \in (0, \infty)$.

Propositions 2 and 4 imply that the cumulative resource requirement function $\mathbf{m}$ must be of one of three types as depicted in Fig. 2. Note that (i) if $C^*(k_0) = \emptyset$, then $\inf_{c\in D(k)} \mathbf{m}(c, k) = \infty$ and $m^*(k_0) = 0$, and (ii) if $C^*(k_0) \neq \emptyset$, then $\inf_{c\in D(k)} \mathbf{m}(c, k) = 0$ and $m^*(k_0) \in (0, \infty)$ (corresponding to the left panel of Fig. 2) or $m^*(k_0) = \infty$ (corresponding to the center or right panel of Fig. 2).

Efficiency and regularity of the maximin Hartwick path, as stated in Theorem 2, are finally dealt with by Proposition 5.

Proposition 5. *Assume* **A1–A3***, and let* $(k_0, m_0) \in \mathbb{R}^2_{++}$ *be given. If* $m_0 \leqslant m^*(k_0)$*, then the Hartwick path* $(c^c(t), k^c(t), r^c(t))$ *from* k_0 *with* $c^c(t) = c$ *for all* $t \geqslant 0$ *and* $\mathbf{m}(c, k_0) = m_0$ *is efficient. If* $m_0 < m^*(k_0)$*, then the Hartwick path* $(c^c(t), k^c(t), r^c(t))$ *from* k_0 *with* $c^c(t) = c$ *for all* $t \geqslant 0$ *and* $\mathbf{m}(c, k_0) = m_0$ *is a regular maximin path.*

Proof of Theorem 1. Assume **A1–A3**, and let $(k_0, m_0) \in \mathbb{R}^2_{++}$ be given. If $C(k_0, m_0)$ is non-empty, then $\inf_{c\in D(k)} \mathbf{m}(c, k) = 0 < m_0$ by Eq. (8) and Proposition 4. Conversely, if $\inf_{c\in D(k)} \mathbf{m}(c, k) < m_0$, then $C^*(k_0) \neq \emptyset$ and there exists $c \in C^*(k_0)$ such that $\mathbf{m}(c, k_0) < m_0$. By Proposition 1, it is feasible to sustain consumption equal to $c > 0$ by following the Hartwick path $(c^c(t), k^c(t), r^c(t))$ from k_0. □

Proof of Theorem 2. Assume **A1–A3**, and let $(k_0, m_0) \in \mathbb{R}^2_{++}$ be given.

Maximin existence. If $C(k_0, m_0) = \emptyset$, then the trivial Hartwick path $(c^0(t), k^0(t), r^0(t))$ where $c^0(t) = 0$, $k^0(t) = k_0$ and $r^0(t) = 0$ for all $k \geqslant 0$ is maximin.

If $C(k_0, m_0) \neq \emptyset$, then by Eq. (8), $C(k_0, m_0) = \{c \in D(k_0)\colon \mathbf{m}(c, k_0) \leqslant m_0\}$. Hence, $C(k_0, m_0)$ is bounded above, since $\mathbf{m}(\cdot, k_0)$ is strictly increasing (Proposition 2) and convex (Proposition 4) on $C^*(k_0)$. It contains its least upper bound, since by Proposition 2, $c^* \equiv \sup\{c \in D(k_0)\colon \mathbf{m}(c, k_0) \leqslant m_0\}$ is contained in $\{c \in D(k_0)\colon \mathbf{m}(c, k_0) \leqslant m_0\}$. This establishes maximin existence also in this case.

Part (i). If $C(k_0, m_0) = \emptyset$, then the trivial Hartwick path $(c^0(t), k^0(t), r^0(t))$ where $c^0(t) = 0$, $k^0(t) = k_0$ and $r^0(t) = 0$ for all $k \geqslant 0$ is maximin.

If $C(k_0, m_0) \neq \emptyset$, then $\mathbf{m}(c^*, k_0) \leqslant m_0$ since $c^* \in \{c \in D(k_0)\colon \mathbf{m}(c, k_0) \leqslant m_0\}$. It now follows from Proposition 1 that the Hartwick path $(c^*(t), k^*(t), r^*(t))$ from k_0 with $c^*(t) = c^*$ for all $t \geqslant 0$ is feasible and thus a maximin path.

Part (ii). The if-part is a direct consequence of Proposition 5. The only-if part follows since the maximin Hartwick path $(c^*(t), k^*(t), r^*(t))$ from k_0 with $c^*(t) = c^*$ for all $t \geqslant 0$ does not satisfy resource exhaustion if $m^*(k_0) < m_0$. In this case, $\int_0^\infty r^*(t)\,dt = m^*(k_0) < m_0$, and thus the path is feasible, but inefficient.

Part (iii) is a direct consequence of Proposition 5. □

The class of CES functions can be used to illustrate Theorems 1 and 2. In the Cobb–Douglas version of the DHSS model, considered by Solow [19], the production function is given by (3), and thus, clearly satisfies assumptions **A1**–**A3**. It is easy to check that $D = \mathbb{R}^2_{++}$ and $D(k) = (0, \infty)$ for each $k \in \mathbb{R}_{++}$. Thus, $\mathbf{p}$ and $\mathbf{r}$ are functions from $\mathbb{R}^2_{++}$ to $\mathbb{R}_{++}$, and it is straightforward to verify that:

$$\mathbf{p}(c,k) = \frac{c^{(1-b)/b}}{b(1-b)^{(1-b)/b}} \cdot \frac{1}{k^{(a/b)}}.$$

Therefore, $\mathbf{m}(c, k_0) = \int_0^\infty \mathbf{p}(c, x)\,dx$ is finite if and only if $a > b$, and $\inf_{c>0} \mathbf{m}(c, k_0)$ equals 0 if $a > b$ and ∞ otherwise. By Theorem 1, $C(k_0, m_0)$ is non-empty if and only if $a > b$, which is the result of Solow [19, Sect. 8 & App. B].

Following Dasgupta and Heal [9, Sect. 7.2] by considering the class of CES production functions beyond the Cobb–Douglas case, so that

$$F(k,r) = \left(ak^{\frac{\sigma-1}{\sigma}} + br^{\frac{\sigma-1}{\sigma}} + (1-a-b)\right)^{\frac{\sigma}{\sigma-1}}$$

for $(k, r) \in \mathbb{R}^2_+$, with $a > 0$, $b > 0$, $a + b \leqslant 1$ and $\sigma > 0$, $\sigma \neq 1$, we have that:

- With $\sigma < 1$, assumptions **A1**–**A3** are satisfied, but $\inf_{c \in D(k)} \mathbf{m}(c, k) = \infty$ since
$$\mathbf{p}(c,k) = \frac{\mathbf{r}(c,k)}{F(k, \mathbf{r}(c,k)) - c} \geqslant \frac{\mathbf{r}(c,k)}{F(k, \mathbf{r}(c,k))} \geqslant b^{\frac{\sigma}{1-\sigma}}$$
if $c \in D(k_0)$. This confirms the well-known result that $C(k_0, m_0)$ is empty.
- With $\sigma > 1$, assumption **A1** is not satisfied—so that Theorem 1 does not apply—while clearly $C(k_0, m_0)$ is non-empty. Problem (4) and, thus, Theorem 1 have no relevance, as $F(k, r) = c$ along an eventual part of a maximin path.

Finally, the limiting case where inputs are perfect substitutes:

$$F(k,r) = k + r \quad \text{for } (k,r) \in \mathbb{R}^2_+, \tag{9}$$

yields an example where $C(k_0, m_0)$ is non-empty and bounded, but does not contain its least upper bound, thereby showing the significance of Theorem 2. Let $(k_0, m_0) = (1, 1)$ be the vector of initial stocks of capital and resource. Even though assumption **A2** and most of assumption **A3** hold, both **A1** (since $F(k, r) = 0$ requires that both inputs are zero) and the last part of **A3** (since $F_{22} = 0$) are violated. Hence, maximin existence is not guaranteed by Theorem 2. And indeed, as demonstrated in online appendix B, $C(1, 1) = (0, 2)$, implying that any consumption level below 2 can be sustained indefinitely. However, since the resource stock cannot be instantaneously transformed into capital, it is not feasible to maintain a level of consumption that never falls below 2. Thus, there is no maximin path in this model.

By Theorem 2(ii), the maximin Hartwick path will not be resource exhausting and thus not efficient if $0 < m^*(k_0) < m_0$ (illustrated by the left panel of Fig. 2). We have not been able to rule out this case without imposing further assumptions beyond **A1**–**A3**. Our investigations indicate that this case might occur under **A1–A3** if the sum of the output elasticities of k and r is not bounded away from zero.

4. Proofs of propositions

Throughout this section we assume **A1–A3**, and let $(k_0, m_0) \in \mathbb{R}^2_{++}$ be given. To prove Proposition 1 we first provide two lemmas.

Lemma 2. *Let $(c, k_0) \in D$. Suppose that there is a solution $k^c(t)$ to the initial value problem (6) for $t \in [0, T)$, for some $T > 0$. Then $\dot{k}^c(t) > 0$ for all $t \in [0, T)$.*

Remark 2. Under the hypothesis of Lemma 2, we have $k^c(t)$ monotonically increasing on $[0, T)$ by the Mean Value Theorem. Then, either $k^c(t)$ is bounded above on $[0, T)$, in which case a finite limit, $\lim_{t\to T} k^c(t)$, exists; or, $k^c(t)$ is not bounded above on $[0, T)$, in which case $k^c(t) \to \infty$ as $t \to T$. In either case, we define

$$k^c(T) = \lim_{t\to T} k^c(t),$$

it being understood that the limit above belongs to $(k_0, \infty]$.

Lemma 3. *Let $(c, k_0) \in D$. Suppose that there is a solution $k^c(t)$ to the initial value problem (6) for $t \in [0, T)$, for some $T > 0$. Then, for every $T' \in (0, T)$,*

$$\int_0^{T'} \mathbf{r}\bigl(c, k^c(t)\bigr)\, dt = \int_{k_0}^{k^c(T')} \mathbf{p}(c, x)\, dx. \tag{10}$$

Furthermore, if:

$$\infty > S \equiv \lim_{T'\to T} \int_0^{T'} \mathbf{r}\bigl(c, k^c(t)\bigr)\, dt, \tag{11}$$

then $k^c(T) < \infty$.

Eq. (10) follows from the change of variables formula (cf. (7)).

Proof of Proposition 1. Assume $c \in C^*(k_0)$ so that $\mathbf{m}(c, k_0) < \infty$. Since $C^*(k_0) \subseteq D(k_0)$, it follows that $c \in D(k_0)$ and there is $\varepsilon \in (0, k_0)$ such that $c \in D(k)$ for all $k \in (k_0 - \varepsilon, \infty)$. Then $\mathbf{r}(c, k)$ is a continuously differentiable function of k from the open set $(k_0 - \varepsilon, \infty)$ to $\mathbb{R}_{++}$. Thus, f defined by

$$f(k) = F\bigl(k, \mathbf{r}(c, k)\bigr) - c \quad \text{for all } k \in (k_0 - \varepsilon, \infty), \tag{12}$$

is a continuously differentiable function of k from the open set $(k_0 - \varepsilon, \infty)$ to $\mathbb{R}$.

By [13, pp. 162–163 & p. 171] there are a *maximal right interval* $[0, \beta)$ and a solution $k^c : [0, \beta) \to (k_0 - \varepsilon, \infty)$ to the initial value problem (6) for $t \in [0, \beta)$, such that if $x^c : [0, \alpha) \to (k_0 - \varepsilon, \infty)$ is a solution to the initial value problem (6) for $t \in [0, \alpha)$, then $\alpha \leqslant \beta$ and $x^c(t) = k^c(t)$ for all $t \in [0, \alpha)$. By Lemma 2, $\dot{k}^c(t) > 0$ and $k^c(t) \in [k_0, \infty)$ for all $t \in [0, \beta)$.

We now claim that

$$\beta = \infty. \tag{13}$$

For all $T' \in (0, \beta)$, using (10) and the definition of the function $\mathbf{m}$, we have:

$$\int_0^{T'} \mathbf{r}\big(c, k^c(t)\big)\,dt = \int_{k_0}^{k^c(T')} \mathbf{p}(c, x)\,dx \leqslant \int_{k_0}^{\infty} \mathbf{p}(c, x)\,dx = \mathbf{m}(c, k_0). \tag{14}$$

Thus, $\infty > \mathbf{m}(c, k_0) \geqslant S \equiv \lim_{T' \to \beta} \int_0^{T'} \mathbf{r}(c, k^c(t))\,dt$ and by Lemma 3, we have $k^c(\beta) < \infty$, where $k^c(\beta) \equiv \lim_{t\to\beta} k^c(t)$. Then by using the theorem in [13, p. 171], claim (13) is established.

Given (13), we know that k^c from $[0, \infty)$ to $(k_0 - \varepsilon, \infty)$ is a solution to the initial value problem (6) for $t \in [0, \infty)$. By defining $c^c(t) = c$ and $r^c(t) = \mathbf{r}(c, k^c(t))$ for $t \in [0, \infty)$, it follows from (14) that $(c^c(t), k^c(t), r^c(t))$ is a path from k_0 with $\int_0^{\infty} r^c(t)\,dt = \mathbf{m}(c, k_0)$. □

Proof of Proposition 2. By Lemma 1, the function $\mathbf{p}$ is continuously differentiable on D with $\mathbf{p}_1(c, k) > 0$ for all $(c, k) \in D$. Hence, if $c' \in C^*(k_0)$ $(\subseteq D(k_0))$ so that $\mathbf{m}(c', k_0) = \int_{k_0}^{\infty} \mathbf{p}(c', x)\,dx < \infty$, then, for all $c \in (0, c')$, $c \in D(k_0)$ and

$$\mathbf{m}(c, k_0) = \int_{k_0}^{\infty} \mathbf{p}(c, x)\,dx < \int_{k_0}^{\infty} \mathbf{p}\big(c', x\big)\,dx < \infty,$$

implying that $c \in C^*(k_0)$ and $\mathbf{m}(\cdot, k_0)$ is strictly increasing on $C^*(k_0)$ if $C^*(k_0) \neq \emptyset$.

Let $c' > 0$ and assume there is $m < \infty$ such that $\mathbf{m}(c, k_0) \leqslant m$ for all $c \in (0, c')$. Suppose that there is $k_1 > k_0$ such that $\int_{k_0}^{k_1} \mathbf{p}(c', x)\,dx > m$. Since, by Lemma 1, there are $c'' > c'$ and $k'' < k_0$ such that the function $\mathbf{p}$ is continuous on $(0, c'') \times (k'', \infty)$, we have that $J(c) \equiv \int_{k_0}^{k_1} \mathbf{p}(c, x)\,dx$ is continuous on the interval $[c'/2, c']$ (see [1, p. 166]). In particular, there is $\tilde{c} \in (0, c')$ such that

$$\mathbf{m}(\tilde{c}, k_0) \geqslant \int_{k_0}^{k_1} \mathbf{p}(\tilde{c}, x)\,dx > m,$$

contradicting the fact that $\mathbf{m}(c, k_0) \leqslant m < \infty$ for all $c \in (0, c')$. This implies that $\mathbf{m}(c', k_0) = \lim_{k_1\to\infty} \int_{k_0}^{k_1} \mathbf{p}(c', x)\,dx \leqslant m$ and thus $c' \in C^*(k_0)$.

It remains to be shown that $\mathbf{m}(\cdot, k_0)$ is continuous on $\operatorname{int}(C^*(k_0))$. Let $c' \in \operatorname{int}(C^*(k_0))$. For any $\varepsilon > 0$, we can choose $c'' \in (c', \sup C^*(k_0))$ and $k_1 > k_0$ such that $\int_{k_1}^{\infty} \mathbf{p}(c'', x)\,dx \leqslant \varepsilon/2$. Since $J(c)$ is continuous on the interval $[c'/2, (c' + c'')/2]$ (see [1, p. 166]), we can choose $\delta \in (0, \min\{c'/2, (c'' - c')/2\})$ such that $|J(c) - J(c')| < \varepsilon/2$ if $|c - c'| < \delta$. Then, for all c satisfying $|c - c'| < \delta$,

$$\big|\mathbf{m}(c, k_0) - \mathbf{m}\big(c', k_0\big)\big| \leqslant \big|J(c) - J\big(c'\big)\big| + \int_{k_1}^{\infty} \big|\mathbf{p}(c, x) - \mathbf{p}\big(c', x\big)\big|\,dx < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

since $\mathbf{p}(\cdot, k_0)$ is increasing and $\max\{c, c'\} < c''$. □

To prove Proposition 3 we first provide two lemmas. A path $(c(t), k(t), r(t))$ from k_0 is *interior* if $k(t) > 0$ and $r(t) > 0$ for all $t \geqslant 0$. The phase diagram argument illustrated in Fig. 1

is based on paths in (k, m) space where the stock of augmentable capital k is a function of the remaining resource stock m. Interior paths have this property, while non-interior paths where resource extraction is zero at some time—so that consumption comes from disinvestment of augmentable capital—do not. This motivates the following lemma.

Lemma 4. *Assume that $C(k_0, m_0) \neq \emptyset$ and let $c' \in C(k_0, m_0)$. If $c \in (0, c')$, then there is a feasible interior path $(c(t), k(t), r(t))$ from (k_0, m_0) with $c(t) > c$ for all $t \geqslant 0$.*

We also establish that the finite resource stock implies that positive consumption cannot be sustained if the stock of augmentable capital is bounded above.

Lemma 5. *If a path $(c(t), k(t), r(t))$ from (k_0, m_0) has the property that there is $c > 0$ such that $c(t) \geqslant c$ for $t \geqslant 0$, then $\limsup_{t\to\infty} k(t) = \infty$.*

Proof of Proposition 3. Assume that there exists a feasible path $(c'(t), k'(t), r'(t))$ from (k_0, m_0) with $c'(t) \geqslant c' > 0$ for $t \geqslant 0$. For each $c \in (0, c')$, fix the feasible interior path $(c(t), k(t), r(t))$ from (k_0, m_0) with $c(t) > c > 0$ for $t \geqslant 0$ established in Lemma 4. By Lemma 5, $\limsup_{t\to\infty} k(t) = \infty$. Let $T_0 = \inf\{t \geqslant 0\colon\ k(t) > k_0\}$. Then $k(T_0) = k_0$ and $\dot{k}(T_0) \geqslant 0$ so that

$$F\bigl(k_0, r(T_0)\bigr) = \dot{k}(T_0) + c(T_0) \geqslant c(T_0) > c,$$

establishing that $(k_0, c) \in D$ and $c \in D(k_0)$. So, there is $\varepsilon \in (0, k_0)$ such that $c \in D(k)$ for all $k \in (k_0 - \varepsilon, \infty)$. Then $\mathbf{r}(c, \cdot)$ is a continuously differentiable function from the open set $(k_0 - \varepsilon, \infty)$ to $\mathbb{R}_{++}$. Thus, f defined by (12) is a continuously differentiable function of k from the open set $(k_0 - \varepsilon, \infty)$ to $\mathbb{R}$. By [13, pp. 162–163 & p. 171] there are a maximal right interval $[0, \beta)$ and a unique solution $k^c : [0, \beta) \to (k_0 - \varepsilon, \infty)$ to the initial value problem (6) for $t \in [0, \beta)$. By Lemma 2, $\dot{k}^c(t) > 0$ and $k^c(t) \in [k_0, \infty)$ for all $t \in [0, \beta)$.

We now proceed to verify that $\mathbf{m}(c, k_0) \leqslant m_0$. It is sufficient to establish that $\int_{k_0}^{k_1} \mathbf{p}(c, x)\, dx < m_0$ for all $k_1 > k_0$. Suppose on the contrary that there is $k_1' \in (k_0, \infty)$ such that $\int_{k_0}^{k_1'} \mathbf{p}(c, x)\, dx \geqslant m_0$. Then, there is $k_1 \in (k_0, k_1']$ such that $\int_{k_0}^{k_1} \mathbf{p}(c, x)\, dx = m_0$. We claim now that

$$k^c(t) > k_1 \quad \text{for some } t \in [0, \beta). \tag{15}$$

If $\beta < \infty$, then this follows directly from the theorem of [13, p. 171]. If $\beta = \infty$, and claim (15) does not hold, then $k^c(t) \leqslant k_1$ for all $t \geqslant 0$. By Lemma 3, we have

$$\int_0^\infty \mathbf{r}\bigl(c, k^c(t)\bigr)\, dt = \int_{k_0}^{k_\infty^c} \mathbf{p}(c, x)\, dx \leqslant \int_{k_0}^{k_1} \mathbf{p}(c, x)\, dx = m_0,$$

where $k_\infty^c \equiv \lim_{t\to\infty} k^c(t)$. However, this means that $(c, k^c(t), \mathbf{r}(c, k^c(t)))$ is a feasible path from (k_0, m_0). Since $c > 0$, it follows from Lemma 5 that $\limsup_{t\to\infty} k^c(t) = \infty$. This clearly contradicts the hypothesis that claim (15) does not hold. Thus, in either case, claim (15) is valid. Using (15), we infer that there is $T^* \in (0, \beta)$ such that $k^c(T^*) = k_1$ and so by Lemma 3,

$$\int_0^{T^*} \mathbf{r}\bigl(c, k^c(t)\bigr)\, dt = \int_{k_0}^{k_1} \mathbf{p}(c, x)\, dx = m_0.$$

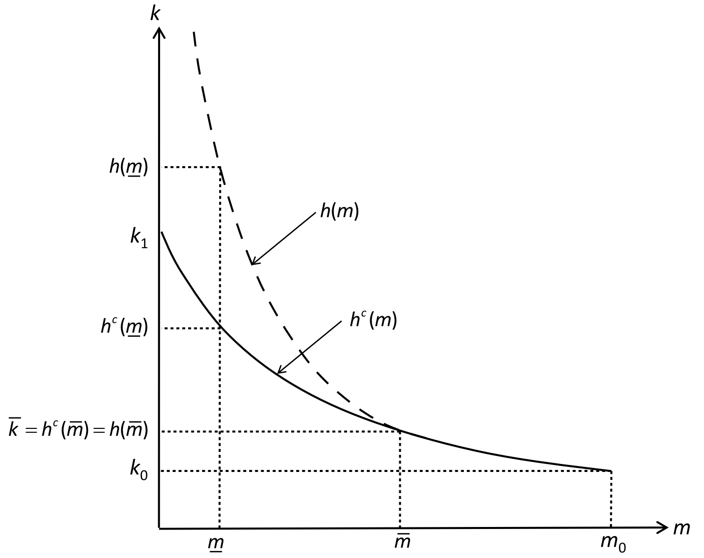

Fig. 3. Illustration of the proof of Proposition 3.

Write $r^c(t) = \mathbf{r}(c, k^c(t))$ and $m^c(t) = m_0 - \int_0^t r^c(\tau)\,d\tau$ for $t \in [0, T^*]$. Since $m^c : [0, T^*] \to [0, m_0]$ is continuously differentiable and decreasing on $[0, T^*]$, it has an inverse function $i^c : [0, m_0] \to [0, T^*]$ which is continuously differentiable and decreasing on $[0, m_0]$. We define $h^c : [0, m_0] \to [k_0, k_1]$ by $h^c(m) = k^c(i^c(m))$. Then h^c is a continuously differentiable and decreasing function on $[0, m_0]$ which determines the stock of augmentable capital as a function of the remaining resource stock for a solution to the initial value problem (6).

As $(c(t), k(t), r(t))$ from (k_0, m_0) is interior, $m : [0, \infty) \to (0, m_0]$ is continuously differentiable and decreasing on $[0, \infty)$ and has an inverse function $i : (0, m_0] \to [0, \infty)$ which is continuously differentiable and decreasing on $(0, m_0]$. We define $h : (0, m_0] \to [k_0, \infty)$ by $h(m) = k(i(m))$. Then h is a continuously differentiable function on $(0, m_0]$ (but not necessarily a decreasing function as $\dot{k}(t)$ is not necessarily positive) which determines the stock of augmentable capital as a function of the remaining resource stock when $(c(t), k(t), r(t))$ is followed.

By Lemma 5 there is $T' \in (0, \infty)$ such that $k(T') > k_1$. Hence,

$$h(\underline{m}) = k\big(T'\big) > k_1 = k^c\big(T^*\big) = h^c(0) > h^c(\underline{m}) \quad \text{where } \underline{m} = m\big(T'\big) \in (0, m_0),$$

while $h(m_0) = k_0 = h^c(m_0)$ so that $\{m \in [\underline{m}, m_0]\colon\ h(m) \leqslant h^c(m)\}$ is non-empty. By continuity of h and h^c on $(0, m_0]$, $\bar{m} = \inf\{m \in [\underline{m}, m_0]\colon\ h(m) \leqslant h^c(m)\} > \underline{m}$ and $h(\bar{m}) = h^c(\bar{m})$; let $\bar{k}$ denote this common value. This is illustrated in Fig. 3.

Since $h(m) > h^c(m)$ for $m \in (\underline{m}, \bar{m})$ and $h(\bar{m}) = h^c(\bar{m})$, we have

$$\frac{h(m) - h(\bar{m})}{m - \bar{m}} < \frac{h^c(m) - h^c(\bar{m})}{m - \bar{m}}$$

for all $m \in (\underline{m}, \bar{m})$. Letting $m \to \bar{m}$ and noting that h and h^c are continuously differentiable on $(0, m_0]$, we obtain $h'(\bar{m}) \leqslant h^{c\prime}(\bar{m})$.

Since $\bar{m} \in [0, m_0)$, there is a unique T such that $m(T) = \bar{m}$ and

$$h'(\bar{m})\dot{m}(T) = h'\big(m(T)\big)\dot{m}(T) = \dot{k}(T).$$

As $\dot{m}(T) = -r(T) < 0$ and $\dot{k}(T) = F(\bar{k}, r(T)) - c(T)$, we have that

$$-h'(\bar{m}) = \frac{\dot{k}(T)}{-\dot{m}(T)} = \frac{F(\bar{k}, r(T)) - c(T)}{r(T)}. \tag{16}$$

Since $\bar{m} \in [0, m_0)$, there is a unique T^c such that $m^c(T^c) = \bar{m}$ and

$$h^{c\prime}(\bar{m})\dot{m}^c(T^c) = h^{c\prime}(m^c(T^c))\dot{m}^c(T^c) = \dot{k}^c(T^c).$$

As $\dot{m}^c(T^c) = -r^c(T^c) < 0$ and $\dot{k}^c(T) = F(\bar{k}, r^c(T)) - c$, we have that

$$-h^{c\prime}(\bar{m}) = \frac{\dot{k}^c(T^c)}{-\dot{m}^c(T^c)} = \frac{F(\bar{k}, r^c(T^c)) - c}{r^c(T^c)}. \tag{17}$$

Recall that $c(T) > c$. Hence, it follows from Lemma 1 that

$$\frac{F(\bar{k}, r(T)) - c(T)}{r(T)} < \frac{F(\bar{k}, r(T)) - c}{r(T)} \leqslant \frac{F(\bar{k}, \mathbf{r}(c, \bar{k})) - c}{\mathbf{r}(c, \bar{k})} = \frac{F(\bar{k}, r^c(T^c)) - c}{r^c(T^c)}.$$

Combined with (16) and (17) this contradicts $h'(\bar{m}) \leqslant h^{c\prime}(\bar{m})$. Thus, $\mathbf{m}(c, k_0) \leqslant m_0$.

It now follows from Proposition 2 that $\mathbf{m}(c', k_0) \leqslant m_0$ since $\mathbf{m}(c, k_0) \leqslant m_0$ for all $c \in (0, c')$. □

Proof of Proposition 4. Assume that $C^*(k_0) \neq \emptyset$.

Part 1: $\mathbf{m}(\cdot, k_0)$ *is convex on* $C^*(k_0)$. We have to prove that for all $c', c'' \in C^*(k_0)$ and $\lambda \in [0, 1]$ we have $\mathbf{m}(\lambda c' + (1 - \lambda)c'', k_0) \leqslant \lambda \mathbf{m}(c', k_0) + (1 - \lambda)\mathbf{m}(c'', k_0)$. This is trivially true for $\lambda = 0$, $\lambda = 1$, or $c' = c''$. So assume $c' \neq c''$ and $\lambda \in (0, 1)$.

By Proposition 1, there are Hartwick paths $(c', k'(t), r'(t))$ and $(c'', k''(t), r''(t))$ from k_0 with $\int_0^\infty r'(t)\,dt = \mathbf{m}(c', k_0)$ and $\int_0^\infty r''(t)\,dt = \mathbf{m}(c'', k_0)$, where suppressed time variables indicate constant consumption. Construct $(c(t), k(t), r(t))$ as follows:

$$c(t) = F(k(t), r(t)) - \dot{k}(t) \qquad \text{for all } t \geqslant 0,$$

$$k(t) = \lambda k'(t) + (1 - \lambda)k''(t) \qquad \text{for all } t \geqslant 0, \tag{18}$$

$$r(t) = \lambda r'(t) + (1 - \lambda)r''(t) \qquad \text{for all } t \geqslant 0. \tag{19}$$

By Proposition 3, it suffices to show that $(c(t), k(t), r(t))$ is feasible from (k_0, m_0), where $m_0 \equiv \lambda \mathbf{m}(c', k_0) + (1 - \lambda)\mathbf{m}(c'', k_0)$, with $c(t) \geqslant \lambda c' + (1 - \lambda)c''$ for all $t \geqslant 0$.

Clearly, $k(t)$ is a differentiable function of t, with $\dot{k}(t) = \lambda \dot{k}'(t) + (1 - \lambda)\dot{k}''(t)$ for $t \geqslant 0$, and $(c(t), r(t))$ are continuous functions of t. Using (18) and (19), for $t \geqslant 0$,

$$\begin{aligned} c(t) &= F(k(t), r(t)) - \dot{k}(t) \\ &\geqslant \lambda\big(F(k'(t), r'(t)) - \dot{k}'(t)\big) + (1 - \lambda)\big(F(k''(t), r''(t)) - \dot{k}''(t)\big) = \lambda c' + (1 - \lambda)c'' \end{aligned}$$

by the concavity of F. Also, $k(t) = \lambda k'(t) + (1 - \lambda)k''(t) > 0$ for $t \geqslant 0$, and (1) is satisfied since $k(0) = \lambda k_0 + (1 - \lambda)k_0 = k_0$. Finally, $\int_0^\infty r(t)\,dt = \lambda \int_0^\infty r'(t)\,dt + (1 - \lambda)\int_0^\infty r''(t)\,dt = \lambda \mathbf{m}(c', k_0) + (1 - \lambda)\mathbf{m}(c'', k_0) = m_0$, thereby completing Part 1.

Part 2: *The range of* $\mathbf{m}(\cdot, k_0)$ *is either equal to* $(0, \infty)$ *or equal to* $(0, m^*(k_0)]$ *where* $m^*(k_0) \in (0, \infty)$. By Proposition 2, it suffices to show that $\lim_{c \downarrow 0} \mathbf{m}(c, k_0) = 0$. Let $c' \in C^*(k_0)$ and denote by $(c', k'(t), r'(t))$ the Hartwick path from k_0 with constant consumption equal to c'. Using Proposition 1, $\int_0^\infty r'(t)\,dt = \mathbf{m}(c', k_0)$. Let $\lambda \in (0, 1)$ and construct $(c(t), k(t), r(t))$ as follows:

$$c(t) = F\big(k(t), r(t)\big) - \dot{k}(t) \quad \text{for all } t \geqslant 0,$$
$$k(t) = \lambda k'(t) + (1-\lambda)k_0 \quad \text{for all } t \geqslant 0,$$
$$r(t) = \lambda r'(t) \quad \text{for all } t \geqslant 0.$$

Then $(c(t), k(t), r(t))$ is a feasible path from $(k_0, \lambda \mathbf{m}(c', k_0))$ with, for $t \geqslant 0$,

$$\begin{aligned} c(t) &= F\big(k(t), r(t)\big) - \dot{k}(t) \\ &\geqslant \lambda\big(F\big(k'(t), r'(t)\big) - \dot{k}'(t)\big) + (1-\lambda)\big(F(k_0, 0) - 0\big) = \lambda c' \end{aligned}$$

by repeating the arguments of Part 1 and using the properties that F is concave and $F(k_0, 0) = 0$. By Proposition 3, $\mathbf{m}(\lambda c', k_0) \leqslant \lambda \mathbf{m}(c', k_0)$, thereby establishing that $\lim_{c \downarrow 0} \mathbf{m}(c, k_0) = 0$ by setting $c = \lambda c'$ and letting $\lambda \downarrow 0$. □

We end this section by presenting the proof of Proposition 5. An interior path $(c(t), k(t), r(t))$ from k_0 satisfies *Hotelling's* no-arbitrage *rule* if $r(t)$ is not only continuous but also differentiable and

$$\frac{\dot{F}_2(k(t), r(t))}{F_2(k(t), r(t))} = F_1\big(k(t), r(t)\big) \quad \text{for all } t \geqslant 0. \tag{HoR}$$

A feasible path $(c(t), k(t), r(t))$ from (k_0, m_0) is *competitive* if there are present-value price functions $p(\cdot) : [0, \infty) \to \mathbb{R}_{++}$ and $(q_1(\cdot), q_2(\cdot)) : [0, \infty) \to \mathbb{R}^2$, where $p(t)$ is continuous and $(q_1(t), q_2(t))$ are differentiable, such that, for all $t \geqslant 0$,

$$\big(c(t), k(t), m(t), \dot{k}(t), \dot{m}(t)\big), \quad \text{where } \dot{k}(t) = F\big(k(t), r(t)\big) - c(t) \text{ and } \dot{m}(t) = -r(t),$$

maximizes instantaneous profits $p(t)c' + q_1(t)\dot{k}' + q_2(t)\dot{m}' + \dot{q}_1(t)k' + \dot{q}_2(t)m'$ over all quintuples $(c', k', m', \dot{k}', \dot{m}')$ in the production possibility set Y defined by:

$$Y \equiv \big\{(c, k, m, \dot{k}, \dot{m}) \in \mathbb{R}^3_+ \times \mathbb{R} \times (-\mathbb{R}_+)\colon\ c + \dot{k} \leqslant F\big(k, (-\dot{m})\big)\big\}.$$

Lemma 6. *If a feasible path $(c(t), k(t), r(t))$ from (k_0, m_0) is interior, then it satisfies Hotelling's rule if and only if it is competitive.*

This result is shown by defining, for an interior path $(c(t), k(t), r(t))$,

$$p(t) = \frac{1}{F_2\big(k(t), r(t)\big)} \quad \text{for all } t \geqslant 0 \tag{P}$$

and $q_1(t) = p(t)$, $q_2(t) = 1$ for all $t \geqslant 0$. Here $p(t)$ is the present-value price of consumption and capital accumulation in terms of resource input, which serves as numeraire without specifying the time of extraction, owing to Hotelling's rule.

An interior path $(c(t), k(t), r(t))$ from k_0 satisfying (HoR) satisfies the *capital value transversality condition* if

$$\lim_{t \to \infty} p(t)k(t) = 0. \tag{CVT}$$

Lemma 7. *If an interior and competitive path $(c(t), k(t), r(t))$ from (k_0, m_0) satisfies the capital value transversality condition and is resource exhausting, then it is efficient.*

An interior and competitive path $(c(t), k(t), r(t))$ from (k_0, m_0) is a *regular maximin path* if it[6]

(i) is egalitarian,
(ii) satisfies the capital value transversality condition and resource exhaustion,
(iii) has finite consumption value:

$$\int_0^\infty p(t) < \infty. \tag{FCV}$$

By Lemma 7, any regular maximin path is efficient.

Proof of Proposition 5. *Part 1: If $m_0 \leqslant m^*(k_0)$, then the Hartwick path $(c^c(t), k^c(t), r^c(t))$ from k_0 with $c^c(t) = c$ for all $t \geqslant 0$ and $\mathbf{m}(c, k_0) = m_0$ is efficient.* The existence of $(c^c(t), k^c(t), r^c(t))$ follows from Propositions 1 and 4. Since $(c^c(t), k^c(t), r^c(t))$ is interior and, by Proposition 1, resource exhausting, it follows from Lemmas 6 and 7 that it is sufficient to show that $(c^c(t), k^c(t), r^c(t))$ satisfies (HoR) and (CVT). By invoking [5, Proposition 3] it follows that $(c^c(t), k^c(t), r^c(t))$ satisfies (HoR), as it is interior and egalitarian, satisfies (HaR) for all $t \geqslant 0$, and has the property that $c^c(t)$, $k^c(t)$ and $r^c(t)$ are continuously differentiable.

To show that $(c^c(t), k^c(t), r^c(t))$ satisfies (CVT), note that by Lemma 5 and the fact that $\dot{k}^c(t) = F(k^c(t), r^c(t)) - c > 0$, we have that $\lim_{t\to\infty} k^c(t) = \infty$. Furthermore, $\int_{k_0}^\infty \mathbf{p}(c, x)\,dx = \mathbf{m}(c, k_0) = m_0 < \infty$, implying by (P) that

$$p(t) = \frac{1}{F_2(k^c(t), r^c(t))} = \mathbf{p}\bigl(c, k^c(t)\bigr) \to 0 \quad \text{as } t \to \infty,$$

since $\mathbf{p}_2(c, k) < 0$ for all $(c, k) \in D$, by the definition of $\mathbf{p}$ and the envelope theorem.

Let $\varepsilon > 0$ be given. By Proposition 1, there is $T_1 > 0$ such that:

$$\bigl|m^c(t) - \mathbf{m}(c, k_0)\bigr| < \frac{\varepsilon}{3} \quad \text{for all } t > T_1. \tag{20}$$

Furthermore, because $\lim_{t\to\infty} p(t) = 0$ there is $T_2 > T_1$ such that:

$$p(t)k^c(T_1) < \frac{\varepsilon}{3} \quad \text{for all } t > T_2. \tag{21}$$

Fix any $T' > T_2$. Then, since $\dot{k}^c(t) > 0$ for all $t \geqslant 0$, and $p(t)$ is positive and decreasing in t, we can use (HaR) to write:

$$\int_{T_1}^{T'} r^c(t)\,dt = \int_{T_1}^{T'} p(t)\dot{k}^c(t)\,dt \geqslant \int_{T_1}^{T'} p\bigl(T'\bigr)\dot{k}^c(t)\,dt$$
$$= p\bigl(T'\bigr)\int_{T_1}^{T'} \dot{k}^c(t)\,dt = p\bigl(T'\bigr)\bigl[k^c\bigl(T'\bigr) - k^c(T_1)\bigr]. \tag{22}$$

6 Dixit et al. [11, p. 553] define a regular maximin path through their conditions (a)–(c). In the context of our one-consumption good model, (i) and (iii) correspond to their conditions (a) and (b), while (ii) is equivalent to their condition (c), given that $q_2(t) = 1$ for all $t \geqslant 0$.

On the other hand, by using (20),

$$\int_{T_1}^{T'} r^c(t)\,dt = m^c(T_1) - m^c(T') = \big(m^c(T_1) - \mathbf{m}(c,k_0)\big) - \big(m^c(T') - \mathbf{m}(c,k_0)\big)$$
$$\leqslant \big|m^c(T_1) - \mathbf{m}(c,k_0)\big| + \big|m^c(T') - \mathbf{m}(c,k_0)\big| < \frac{2\varepsilon}{3}. \tag{23}$$

Combining (22) and (23),

$$p(t)\big[k^c(t) - k^c(T_1)\big] < \frac{2\varepsilon}{3} \quad \text{for all } t > T_2.$$

This yields:

$$p(t)k^c(t) < p(t)k^c(T_1) + \frac{2\varepsilon}{3} \quad \text{for all } t > T_2. \tag{24}$$

By (21) and (24) we obtain $p(t)k^c(t) < \varepsilon$ for all $t > T_2$, thus establishing (CVT).

Part 2: *If $m_0 < m^*(k_0)$, then the Hartwick path $(c^c(t), k^c(t), r^c(t))$ from k_0 with $c^c(t) = c$ for all $t \geqslant 0$ and $\mathbf{m}(c,k_0) = m_0$ is a regular maximin path.* By Part 1 and the definition of a regular maximin path, it is sufficient to show that $(c^c(t), k^c(t), r^c(t))$ satisfies (FCV).

By Propositions 1 and 4, if $m_0 < m^*(k_0)$, then there are $c' > c$ and a Hartwick path $(c^{c'}(t), k^{c'}(t), r^{c'}(t))$ from k_0 with $c^{c'}(t) = c'$ for all $t \geqslant 0$ and $m'_0 \equiv \mathbf{m}(c',k_0) > m_0$. By Part 1, $(c^c(t), k^c(t), r^c(t))$ satisfies (HoR) and is thus, by Lemma 6, competitive at prices $p(t) = \mathbf{p}(c, k^c(t))$, $q_1(t) = p(t)$ and $q_2(t) = 1$. Therefore:

$$\int_0^T p(t)\big(c' - c\big)\,dt \leqslant \int_0^T \frac{d}{dt}\big(p(t)\big(k^c(t) - k^{c'}(t)\big)\big)\,dt + \int_0^T \big(\dot{m}^c(t) - \dot{m}^{c'}(t)\big)\,dt$$
$$= p(T)\big(k^c(T) - k^{c'}(T)\big) + \int_0^T \big(r^{c'}(t) - r^c(t)\big)\,dt.$$

Since $p(T)k^{c'}(T) \geqslant 0$ for all $T \geqslant 0$ and, by Part 1, $(c^c(t), k^c(t), r^c(t))$ satisfies (CVT):

$$\int_0^\infty p(t)\big(c' - c\big)\,dt \leqslant \int_0^\infty \big(r^{c'}(t) - r^c(t)\big)\,dt = m'_0 - m_0.$$

Hence, $\int_0^\infty p(t)\,dt \leqslant (m'_0 - m_0)/(c' - c) < \infty$, thereby establishing (FCV). □

5. Concluding remark

Based on an alternative interpretation of Hartwick's rule in the continuous DHSS-model we have shown that following this rule will lead to a minimization of cumulative resource input along constant consumption paths. Using this insight we have derived a new necessary and sufficient condition for sustaining consumption indefinitely bounded away from zero. Based on this characterization, we have established the existence and efficiency of maximin paths under much weaker conditions than before. The approach which we have used to obtain these results, however, cannot be applied to paths with non-constant consumption over time. As established in earlier work [5], along an optimal path where Hotelling's rule is fulfilled, obeying Hartwick's rule (and thus minimizing resource input per unit of capital accumulation) is not compatible with having consumption varying over time.

The following result [16, Theorem 1] yields intuition[7]: If an interior and competitive path $(c(t), k(t), r(t))$ from (k_0, m_0) is efficient, then any feasible path $(c'(\tau), k'(\tau), r'(\tau))$ from (k', m') with $c'(\tau) \geqslant c(T + \tau)$ for all $\tau \in [0, \infty)$ must satisfy

$$p(T)k' + m' \geqslant p(T)k(T) + m(T),$$

where $p(T) = 1/F_2(k(T), r(T))$. Therefore, along an efficient path where $c(t)$ is constant, $p(T)k(t) + m(t)$ is minimized at T among all t in the neighborhood of T, implying as a necessary first-order condition that the derivative of $p(T)k(t) + m(t)$ w.r.t. t is zero at T, leading to Hartwick's rule: $p(T)\dot{k}(T) = -\dot{m}(T) = r(T)$.

By the same reasoning, if $c(t)$ is strictly increasing, so that $c'(\tau)$ defined by $c'(\tau) = c(T+\tau)$ is increasing in T for each $\tau \in [0, \infty)$, then $p(T)k(t)+m(t)$ is increasing as a function of t at T and $p(T)\dot{k}(T) > -\dot{m}(T) = r(T)$. This makes it worthwhile to increase the speed of the trajectory in (k, m) space at the expense of its steepness. Hence, at each t, $r(t)$ and $\dot{k}(t) = F(k(t), r(t)) - c(t)$ exceeds the rates at which resource input per unit of capital accumulation would have been minimized given $c(t)$. These results, which are formally demonstrated in online appendix C, show how efficient paths with growing consumption deviate from Hartwick's rule.

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Appendix A. Supplementary material

Supplementary material related to this article can be found online at http://dx.doi.org/10.1016/j.jet.2013.04.019

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